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Adinkras: A Graphical Technology for Supersymmetric Representation Theory[1]

Michael Faux[2]

Department of Physics
Hobart and William Smith College
Geneva, NY 14456, USA

and

S. J. Gates, Jr[3]

Center for String and Particle Theory
Department of Physics, University of Maryland
College Park, MD 20742-4111 USA

ABSTRACT

We present a symbolic method for organizing the representation theory of one-dimensional superalgebras. This relies on special objects, which we have called adinkra symbols, which supply tangible geometric forms to the still-emerging mathematical basis underlying supersymmetry.

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[2] faux@hws.edu
[3] gatess@wam.umd.edu

1   Introduction

There are important examples in which theoretical physics incorporates elegant motifs to represent mathematical conceptions that are vastly simplified thereby. One such example is the wide-spread use of Feynman diagrams. Another one of these is Salam-Strathdee superspace, a stalwart construction which has proven most helpful in organizing fundamental notions in field theory and in string theory. Despite its successes, however, there are vexing limitations which bedevil attempts to use this latter construction to understand certain yet-mysterious aspects of off-shell supersymmetry. This situation would seemingly benefit from an improved organizational scheme. In this paper, we introduce a graphical paradigm which shows some promise in providing a new symbolic technology for usefully re-conceptualizing problems in supersymmetric representation theory.

The use of symbols to connote ideas which defy simple verbalization is perhaps one of the oldest of human traditions. The Asante people of West Africa have long been accustomed to using simple yet elegant motifs known as Adinkra symbols, to serve just this purpose. With a nod to this tradition, we christen our graphical symbols as "Adinkras."

Our focus in this paper pertains most superficially to the classification of off-shell representations of arbitrary N-extended one-dimensional superalgebras. However, for some time, we have been aware of evidence that suggests that *every* superalgebra, in any spacetime dimension, has its representation theory fully encoded in the representations of corresponding one-dimensional superalgebras. This idea, and much of the relevant mathematical technology for substantiating this idea, has been developed in a series of previous papers [1, 2, 3]. One purpose of this current paper is to introduce a new notational tool which, among other things, adds tangible conceptual forms useful for discerning both the content and the ramifications of this mathematics. The tool we introduce is a new sort of symbol which usefully represents supermultiplets.

The relevance of our investigation extends beyond the realm of representation theory, however. Indeed, there are reasons to suppose that supersymmetric quantum mechanics might include undiscovered algebraic structures related to interesting fundamental questions. Consider the simple observation that every quantum field theory formulated in any spacetime dimension, has a corresponding supersymmetric quantum mechanical model obtained by dimensionally reducing all of the spatial dimensions. We refer to these quantum mechanical models as "shadows" of the original quantum field theories. Higher spacetime dimension D manifests in the shadow version as higher N, whereas structure group $SO(D-1,1)$ transformations manifest
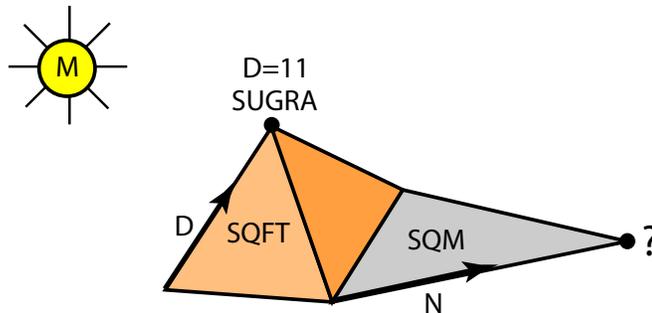
Figure 1: Each supersymmetric quantum field theory has a "shadow" in supersymmetric quantum mechanics obtained by dimensionally reducing all of the spatial dimensions in the field theory.

as R-transformations. Those quantum field theories having remarkable algebraic features, anomaly cancellation for example, must have algebraically-interesting shadows as well. Since eleven-dimensional supergravity is a unique theory, the corresponding $N = 32$ supersymmetric quantum mechanics certainly exhibits its own special uniqueness. One might wonder how the feature of anomaly freedom in effective string theory descriptions of ten-dimensional supergravity manifest on corresponding shadow mechanics. This viewpoint might be useful in discerning whatever analogs of string-theory modular invariance exist in M-theory.

We should emphasize the importance of finding an overarching off-shell representation theory for supersymmetry. This is a problem that has been largely ignored as theoretical physicists have been able to uncover ever more interesting and complicated theories that involve supersymmetry by ever more creative means. We refer to this as the "auxiliary field problem." Some familiar systems in which this problem is observed are 11D supergravity and all known 10D supersymmetric systems. Since each of these particular systems are special limits of closely related M-theory and 10D superstrings, it follows that any increase in our understanding of these special limits is likely to accrue benefits to our understanding of the full theories.

This paper is structured as follows.

In sections 2 through 7 we present an overview of the mathematical basis for the core part of the paper, which begins in section 8. The review sections are included in part to make this paper relatively self-contained. But these also include several important new definitions and include commentary which may prove helpful to the reader. In these sections we describe the elemental superalgebra and set our notational conventions. We review that class of irreducible representations which includes gen-

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eralized scalar and generalized spinor multiplets, and discuss aspects of automorphic duality transformations. We review the connection between the multiplets mentioned above and the algebras $\mathcal{GR}(d, N)$. We review the connection between the representations of $\mathcal{GR}(d, N)$ and those of the Clifford algebras $C(N, 1)$. We then review the notion of a root superfield proposed in [1] which may provide the mathematical lynch pin for the classification of all supermultiplets.

In sections 8 through 14 we methodically develop the conception of adinkra symbols referred to above. In successive sections, we show how elemental $N = 1$ adinkra symbols can be combined to describe higher-N representations, and how duality maps connect these with adinkras describing distinct multiplets. We show how the adinkra symbols fit naturally into the concept of a root space and how supersymmetry transformations can be viewed in terms of flows on a lattice. We use these techniques to describe new multiplets which exhibit interesting topological distinctions. We use these techniques to comprehensively describe all of the known irreducible multiplets for $N \leq 4$, and a few interesting reducible multiplets. In so doing, we are hopeful that the discussion presents a satisfying re-conceptualization of traditional superspace reduction techniques, and a satisfying re-conceptualization of gauge invariance in supermultiplets.

2　The Elemental $d = 1$ Superalgebra

The most basic of all superalgebras is the $d = 1$ $N = 1$ superalgebra, which can be written as

$$[\, \delta_Q(\epsilon_1) \, , \, \delta_Q(\epsilon_2) \,] = -2 \, i \, \epsilon_1 \, \epsilon_2 \, \partial_\tau \, , \tag{2.1}$$

where $\epsilon_{1,2}$ are real anticommuting parameters and τ is a "proper time" which parameterizes the one-dimensional space. We like to interpret this space as the worldline traced out by a particle in an ambient "target-space". There are two irreducible representations of (2.1). The first of these is the $d = 1$ $N = 1$ scalar multiplet, which includes a real commuting field ϕ ($\phi = \phi^*$ where the $*$-operation denotes "superspace conjugation) as lowest component and a real anticommuting field ψ ($\psi = \psi^*$) as highest component. The other basic multiplet is the $d = 1$ $N = 1$ spinor multiplet, which includes a real anticommuting field η as a lowest component and a real commuting field B as highest component. The supersymmetry transformation rules are

$$\begin{aligned}
\delta_Q \, \phi &= i \, \epsilon \, \psi & \delta_Q \, \eta &= \epsilon \, B \\
\delta_Q \, \psi &= \epsilon \, \dot{\phi} & \delta_Q \, B &= i \, \epsilon \, \dot{\eta} \, ,
\end{aligned} \tag{2.2}$$

where ϵ is a real anticommuting parameter ($\epsilon = \epsilon^*$). One way to describe these multiplets is via the superfields,

$$\Phi = \phi + i\,\theta\,\psi \qquad\qquad \Lambda = \eta + \theta\,B\,, \qquad\qquad (2.3)$$

where θ is a real anticommuting coordinate. If we introduce superspace operators

$$Q = i\partial/\partial\theta + \theta\,\partial_\tau$$

$$D = i\partial/\partial\theta - \theta\,\partial_\tau\,, \qquad\qquad (2.4)$$

then the transformation rules (2.2) follow from acting on Φ and Λ with $\delta_Q(\epsilon) = -i\,\epsilon\,Q$. We can write invariant actions as

$$S_\Phi = \int dt\,d\theta \left(\tfrac{1}{2}\,\Phi\,\partial_\tau\,D\,\Phi \right)$$

$$S_\Lambda = \int dt\,d\theta \left(-\tfrac{1}{2}\,i\,\Lambda\,D\,\Lambda \right) \qquad\qquad (2.5)$$

In terms of components, i.e., after performing the θ integrations, there are described by

$$S_\Phi = \int dt \left(\tfrac{1}{2}\,\dot\phi^2 - \tfrac{1}{2}\,i\,\psi\,\dot\psi \right)$$

$$S_\Psi = \int dt \left(-\tfrac{1}{2}\,i\,\eta\,\dot\eta + \tfrac{1}{2}\,B^2 \right). \qquad\qquad (2.6)$$

It is also possible to add a superpotential for Φ by adding $\int dt\,d\theta\,W(\Phi)$ to S_Φ. Other interactions are also possible.

3   Automorphic Duality Transformations

A useful operation which maps between the two irreducible $N = 1$ multiplets was described in [1]. In terms of components, this is realized by making the following replacements

$$(\dot\phi\,,\,\psi) \;\leftrightarrow\; (B\,,\,\eta)\,. \qquad\qquad (3.7)$$

In terms of superfields, these are equivalent to

$$\Lambda \;\leftrightarrow\; -D\,\Phi\,. \qquad\qquad (3.8)$$

Under this operation, the transformation rules for the scalar and spinor multiplets are interchanged and the actions S_Φ and S_Λ are also interchanged. In other words, a map

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(3.7) suffices to replace a scalar multiplet with a spinor multiplet and vice-versa. Since this generates an automorphism on the space of superalgebra representations, this is referred to as an automorphic duality, or an AD map for short. The term duality is used here not in the sense that the multiplets, or theories constructed using these multiplets, are equivalent. Instead, the term implies simply that these constructions are paired by this operation.

The AD map connecting a scalar multiplet with a spinor multiplet is intrinsically non-local. This is because (3.7) implies $\phi(\tau) \to \int dt\, B(\tau)$. However, this is realized in a local way on the transformation rules (4.2) and on the actions (4.4) because ϕ always appears differentiated, i.e., because there is a shift symmetry $\phi \to \phi + c$, where c is a constant parameter. (A superpotential would generally spoil this property.) It is possible to generalize these actions to describe quantum mechanical sigma models. In these cases, the presence of a shift symmetry implies that the target space has an isometry. Interestingly, such isometries are precisely the ingredient needed to couple a background vector field to the theory so as to switch on a supersymmetry central charge [4, 5]. Therefore, the ability to perform automorphic duality transformations is equivalent to the ability to include a central term in the superalgebra. As shown in [4], these charges imply interesting target space dualities similar to T-dualities in string theory. This motivates a basic connection between automorphic duality and non-trivial target space dualities.

The AD map (3.7) describes a quantum mechanical version of Hodge duality. To see this, note that in field theories Hodge duality maps a P-form Ω_P into a $D - P - 2$ form $\tilde{\Omega}_{D-P-2}$ via $d\,\Omega_P \to *\,d\,\tilde{\Omega}_{D-P-2}$. If one starts with a scalar field ϕ in one-dimension, then $D = 1$ and $P = 0$, in which case this implements a map $\phi \to \Omega_{-1}$, where Ω_{-1} is a formal "minus-one"-form, an object whose exterior derivative is a zero-form. This is precisely what characterizes the field B which appears as the image of ϕ under (3.7).

4 Extended Supersymmetry

The N-extended $d = 1$ superalgebra is described by

$$[\,\delta_Q(\epsilon_1^I)\,,\ \delta_Q(\epsilon_2^J)\,] = -2\,i\,\epsilon_1^I\,\epsilon_2^J\,\partial_\tau\,, \tag{4.1}$$

where $I = 1, ..., N$, and ϵ_i^I are a set of real anticommuting parameters. Although it is possible to include a central term on the right-hand side of (4.1), we do not do so

at this time [4]. In this section we review a particular class of minimal representations to (4.1). These generalize the $N = 1$ scalar and spinor multiplets described above. Many other representations exist which lie outside this class, however. Likely, all other representations can be discerned and organized using technology developed in [1]. In this and in the following three sections we briefly review these results, since these provide the mathematical basis behind the core presentation of this paper.

4.1 Scalar Multiplets

One class of representations describes generalized scalar multiplets. The transformation rules are determined by making the following ansatz,

$$\delta_Q \, \phi_i = -i \, \epsilon^I \, (\, L_I \,)_i{}^{\widehat{j}} \, \psi_{\widehat{j}}$$

$$\delta_Q \, \psi_{\widehat{i}} = \epsilon^I \, (\, R_I \,)_{\widehat{i}}{}^{j} \, \dot{\phi}_j \, , \tag{4.2}$$

where $\phi_i(\tau)$ is a set of real commuting fields and $\psi_{\widehat{i}}(\tau)$ is a set of real anticommuting fields. Ordinarily, supersymmetry requires an equal number of bosons and fermions. Accordingly, the indices i and $\widehat{i}$ each have the same multiplicity, denoted d. Accordingly, $i = 1, ..., d$ and $\widehat{i} = 1, ..., d$. Furthermore, since ϕ_i and $\psi_{\widehat{i}}$ are each real, it follows that the matrices $(\, L_I \,)_i{}^{\widehat{j}}$ and $(\, R_I \,)_{\widehat{i}}{}^{j}$ are real. The algebra (4.1) imposes the following restrictions on L_I and R_I,

$$(\, L_J \, R_I + L_I \, R_J \,)_i{}^{j} = -2 \, \delta_{IJ} \, \delta_i{}^{j}$$

$$(\, R_J \, L_I + R_I \, L_J \,)_{\widehat{i}}{}^{\widehat{j}} = -2 \, \delta_{IJ} \, \delta_{\widehat{i}}{}^{\widehat{j}} \, . \tag{4.3}$$

There is no reason from a purely algebraic point of view to impose an *a priori* relationship between L_I and R_I. Nevertheless, a certain minimalist dynamical consideration does imply one more restriction. In particular, we require that the kinetic action described by

$$S_{SM} = \int dt \, \left(\tfrac{1}{2} \, \dot{\phi}^i \, \dot{\phi}_i - \tfrac{1}{2} \, i \, \psi^{\widehat{i}} \, \dot{\psi}_{\widehat{i}} \right) , \tag{4.4}$$

be invariant under the transformations (4.2). In (4.4), indices are raised according to $\phi^i = \delta^{ij} \, \phi_j$ and $\psi^{\widehat{i}} = \delta^{\widehat{i}\widehat{j}} \, \psi_{\widehat{j}}$. The more general case, describing sigma models with a curved target space, involves additional subtlety not addressed in this paper. The action (4.4) is invariant under (4.2) only if

$$(\, L_I^T \,)^{\widehat{i}j} = -(\, R_I \,)^{\widehat{i}j} \, . \tag{4.5}$$

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[4]See Appendix A for a brief discussion pertaining to such central extensions.

This final requirement defines the operator R_I in terms of L_I, or vice versa. Taken together, the three requirements given in (4.3) and (4.5) describe an algebra which has been designated $\mathcal{GR}(d, N)$.

It is possible to define "twisted scalar multiplets" using the alternate transformation rules obtained by interchanging the placement of L_I and R_I in the transformation rules (4.2). In this case, the algebraic requirements on L_I and R_I are identical to those in the untwisted case.

4.2   Spinor Multiplets

Another class of representations describes generalized spinor multiplets. These are determined by analogy to the previous discussion. Accordingly, the transformation rules are determined using the following ansatz

$$\delta_Q \, \eta_{\widehat{i}} \;\; = \;\; \epsilon^I \, (\, R_I \,)_{\widehat{i}}{}^j \, B_j$$

$$\delta_Q \, B_i \;\; = \;\; -i \, \epsilon^I \, (\, L_I \,)_i{}^{\widehat{j}} \, \dot{\eta}_{\widehat{j}}, \tag{4.6}$$

where $\eta_{\widehat{i}}(\tau)$ describes d real anticommuting fields and $F_i(\tau)$ describes d real commuting fields. We require that the transformation rules (4.6) describe the algebra (4.1). We also impose that the minimalist kinetic action, given by

$$S_{FM} = \int dt \left(\, -\tfrac{1}{2} \, i \, \eta^{\widehat{i}} \, \dot{\eta}_{\widehat{i}} + \tfrac{1}{2} \, B^i \, B_i \right), \tag{4.7}$$

be a supersymmetry invariant. Together, these imply precisely the same restrictions on L_I and R_I as given in (4.3) and (4.5).

It is possible to define "twisted spinor multiplets" using the alternate transformation rules obtained by interchanging the placement of L_I and R_I in (4.6). In this case, the algebraic requirements on L_I and R_I are once again identical to those in the untwisted case.

5   An Algebraic Basis for Generalized Superfields

The existence of the supermuliplets described above hinges on the $\mathcal{GR}(d, N)$ algebras, defined by

$$(\, L_I \, R_J + L_J \, R_I \,)_i{}^j \;\; = \;\; -2 \, \delta_{IJ} \, \delta_i{}^j$$

$$(\, R_I \, L_J + R_J \, L_I \,)_{\widehat{i}}{}^{\widehat{j}} \;\; = \;\; -2 \, \delta_{IJ} \, \delta_{\widehat{i}}{}^{\widehat{j}}$$

$$\delta^{\widehat{i}\widehat{k}} \, (\, R_I \,)_{\widehat{k}}{}^j \;\; = \;\; -\delta^{jk} \, (\, L_I \,)_k{}^{\widehat{i}}, \tag{5.1}$$

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where $(L_I)_i{}^{\hat{\jmath}}$ and $(R_I)_{\hat{\imath}}{}^j$ describe two sets of N $d \times d$ matrices. Let hatted indices take values in one vector space $\mathcal{V}_R \cong \mathbb{R}^d$ and let un-hatted indices take values in another vector space $\mathcal{V}_L \cong \mathbb{R}^d$. In this way, the L_I matrices describe linear operators which map elements of $\mathcal{V}_R$ into elements of $\mathcal{V}_L$, and the R_I matrices describe linear operators which map elements of $\mathcal{V}_L$ into elements of $\mathcal{V}_R$. It is useful to define four distinct sets of linear transformations which act on and between the two vector spaces $\mathcal{V}_L$ and $\mathcal{V}_R$ according to

$$\{\mathcal{M}_L\} : \mathcal{V}_R \to \mathcal{V}_L \qquad \{\mathcal{U}_L\} : \mathcal{V}_L \to \mathcal{V}_L$$

$$\{\mathcal{M}_R\} : \mathcal{V}_L \to \mathcal{V}_R \qquad \{\mathcal{U}_R\} : \mathcal{V}_R \to \mathcal{V}_R, \tag{5.2}$$

In this way $(L_I)_i{}^{\hat{\jmath}} \in \{\mathcal{M}_L\}$ and $(R_I)_{\hat{\imath}}{}^j \in \{\mathcal{M}_R\}$. Furthermore, $(L_I R_J)_i{}^j \in \{\mathcal{U}_L\}$ and $(R_I L_J)_{\hat{\imath}}{}^{\hat{\jmath}} \in \{\mathcal{U}_R\}$. Each of the sets described in (5.2) define a vector space in its own right.

For a given value of N, there is a minimal value of d, called d_N, for which N linearly independent real matrices L_I exist which satisfy (5.1). The value d_N gives the number of off-shell bosonic (or fermionic) degrees of freedom in the minimal supersymmetry matter multiplets for that value of N. To determine d_N, notice that there is a unique way to write N in terms of a mod 8 decomposition, $N = 8\,m + n$. Here $m = 0, 1, 2, 3, ...$ counts cycles of 8, and $n = 1, 2, 3, ...$ counts the position in the cycle. For instance $N = 7$ corresponds to $(m, n) = (0, 7)$, $N = 17$ corresponds to $(m, n) = (2, 1)$, and $N = 714$ corresponds to $(m, n) = (89, 2)$. The values of d_N are given by

$$d_N = 16^m\, f_{\mathrm{RH}}(n), \tag{5.3}$$

where $f_{\mathrm{RH}}(n)$ is the so-called Radon-Hurwitz function [8, 1], defined as $f_{\mathrm{RH}}(n) = 2^r$ where r is the nearest integer greater than or equal to $\log_2 n$. The results are tabulated in Table 1. Explicit matrix representations of L_I and R_I are given for $N \leq 8$ in Appendix A of [2]. This is generalized to arbitrary N using a recursive scheme in [3].

The enveloping algebra $\mathcal{EGR}(d, N) \cong \{\mathcal{M}_L\} \oplus \{\mathcal{M}_R\} \oplus \{\mathcal{U}_L\} \oplus \{\mathcal{U}_R\}$ consists of all linear maps on and between $\mathcal{V}_R$ and $\mathcal{V}_L$. Note that $\mathcal{GR}(d, N) \subset \mathcal{EGR}(d, N)$. A subalgebra of $\mathcal{EGR}(d, N)$ is generated by the two sets of p-forms defined as wedge products involving L_I and R_I,

$$\begin{aligned} f_I &= L_I & \tilde{f}_I &= R_I \\ f_{IJ} &= L_{[I} R_{J]} & \tilde{f}_{IJ} &= R_{[I} L_{J]} \\ f_{IJK} &= L_{[I} R_J L_{K]} & \tilde{f}_{IJK} &= R_{[I} L_J R_{K]} \end{aligned} \tag{5.4}$$

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m	n							
	1	2	3	4	5	6	7	8
0	1	2	4	4	8	8	8	8
1	16	32	64	64	128	128	128	128
2	256	512	1024	1024	2048	2048	2048	2048
3	4096	8192	16,384	16,384	32,768	32,768	32,768	32,768
	(etcetera)							
type	N	AC	Q	Q	Q	AC	N	N

Table 1: Values of d_N where $N = 8\,m + n$ for all $N \le 32$. The $m = 0$ row enumerates d_1 through d_8, the $m = 1$ row enumerates d_9 through d_{16}, the $m = 2$ row enumerates d_{17} through d_{24}, and the $m = 3$ row enumerates d_{25} through d_{32}. This table can be continued to include an arbitrary number of rows. The final row indicates the "type" of the $\mathcal{EGR}(d_N, N)$ representations.

and so forth. Each set of p-forms divides into even forms and odd forms, such that

$$ f_{[\text{odd}]} \oplus f_{[\text{even}]} \oplus \tilde{f}_{[\text{odd}]} \oplus \tilde{f}_{[\text{even}]} \in \{\,\mathcal{M}_L\,\} \oplus \{\,\mathcal{U}_L\,\} \oplus \{\,\mathcal{M}_R\,\} \oplus \{\,\mathcal{U}_R\,\} \qquad (5.5) $$

Collectively, these operators generate an algebra denoted $\wedge\mathcal{GR}(d, N)$.

It is generally so that $\wedge\mathcal{GR}(d_N, N) \subset \mathcal{EGR}(d_N, N)$, although for some values of N, it turns out that $\wedge\mathcal{GR}(d_N, N) \cong \mathcal{EGR}(d_N, N)$. In the latter case the algebra $\mathcal{EGR}(d_N, N)$ is said to be normal. Otherwise the algebra falls into one of two classes, known as almost complex or quaternionic, depending on whether $\mathcal{EGR}(d_N, N)$ contains two or four copies of $\mathcal{GR}(d_N, N)$, respectively. In the almost complex case, $\mathcal{EGR}(d_N, N)$ includes an operator, called $\mathcal{D}$ which interconnects the two copies of $\wedge\mathcal{GR}(d_N, N)$. In the case of quaternionic algebras there is a triplet of operators $\mathcal{E}^{1,2,3}$ which interconnect the four copies of $\wedge\mathcal{GR}(d, N)$. In the balance of this paper we concern ourselves with constructions built using the algebras $\wedge\mathcal{GR}(d_N, N)$, rather than $\mathcal{EGR}(d_N, N)$. A consequence is that the operators $\mathcal{D}$ and $\mathcal{E}^\alpha$ will not play a role in this paper. We suspect, however, that the operators $\mathcal{D}$ and $\mathcal{E}^\alpha$ will contribute in an interesting way in a more comprehensive supersymmetry representation theory. At the present time, however, their significance is not yet fully appreciated [5]. We distinguish the vector spaces spanned by $\wedge\mathcal{GR}(d, N)$ by use of a "prime" symbol. For instance, $f_{[\text{odd}]} \in \{\,\mathcal{M}_L\,\}'$. The vector space $\{\,\mathcal{M}_L\,\}'$ may be smaller than $\{\,\mathcal{M}_L\,\}$ in

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[5]One possibility is that these operators are needed to describe multiplets with a central charge.

the case of almost complex or quaternionic algebras. Similar statements pertain to the other three vector spaces defined in (5.2).

In [1, 2, 3] a close connection between the algebras $\mathcal{GR}(d_N, N)$ and $C(N, 1)$ was exploited to describe the representation theory of the former algebra in terms of the representation theory of the latter. This is helpful because Clifford algebra representations have been studied extensively, and are readily available in the literature. The same Clifford algebras play a seemingly different role in describing spinors in higher dimensional field theories. This may imply interesting "shadow" relationships between representations of $D \geq 2$ superalgebras with analogous representations of $d = 1$ superalgebras.

A very brief synopsis of the connection between representations of $\mathcal{GR}(d, N)$ and those of $C(N, 1)$ follows. For a more detailed description, the reader is referred to [1] and references therein. The Clifford algebra $C(N, 1)$ is defined by

$$\{\, \Gamma_{\widehat{I}}, \, \Gamma_{\widehat{J}} \,\} = -2\,\eta_{\widehat{I}\widehat{J}}, \tag{5.6}$$

where $\widehat{I}, \widehat{J} = 1, ..., N + 1$ and $\eta_{\widehat{I}\widehat{J}} = \mathrm{diag}(1, ..., 1, -1)$. For each positive integer N there exists a $2\,d \times 2\,d$ matrix representation to (5.6) such that the first N Gamma matrices $\Gamma_I = \{\, \Gamma_1, ..., \Gamma_N \,\}$ are real and antisymmetric, and where

$$\Gamma_I = \begin{pmatrix} 0 & L_I \\ R_I & 0 \end{pmatrix}. \tag{5.7}$$

The smaller matrices L_I and R_I which appear here are each $d \times d$, and provide a representation of $\mathcal{GR}(d, N)$.

6   Clifford Algebra Superfields

The multiplets reviewed in section 4 arise from a derivation on a superspace $\mathcal{SM} \cong \mathcal{V}_L \oplus \mathcal{V}_R$, where $\mathcal{V}_L$ and $\mathcal{V}_R$ are the vector spaces described above. For instance, in the case of the scalar multiplet, $\phi_i(\tau) \in \mathcal{V}_L$ and $\psi_{\widehat{i}}(\tau) \in \mathcal{V}_R$ are the superfield "components". In this way, the world-line of a superparticle is described by a pair of trajectories, one in $\mathcal{V}_L$ and the other in $\mathcal{V}_R$. There are other possibilities, however.

Consider instead a different superspace defined as $\mathcal{SM}' \cong \mathcal{U}_L \oplus \mathcal{M}_R$. Parameterize $\mathcal{SM}'$ using as component fields

$$\begin{aligned} \Phi_i{}^j(\tau) &\in \{\mathcal{U}_L\}' \\ \Psi_{\widehat{i}}{}^j(\tau) &\in \{\mathcal{M}_R\}'. \end{aligned} \tag{6.1}$$

Therefore, $\Phi_i{}^j(\tau)$ and $\Psi_{\hat{i}}{}^j(\tau)$ describe fields on the particle world-line which take values in these vector spaces. We can expand the fields in terms of the bases $f_{[\text{even}]}$ and $\widehat{f}_{[\text{odd}]}$ as follows,

$$
\Phi_i{}^j = \phi\,\delta_i{}^j + \phi^{IJ}\,(\,f_{IJ}\,)_i{}^j + \phi^{IJKL}\,(\,f_{IJKL}\,)_i{}^j + \cdots
$$

$$
\Psi_{\hat{i}}{}^j = \psi^I\,(\,\widehat{f}_I\,)_{\hat{i}}{}^j + \psi^{IJK}\,(\,\widehat{f}_{IJK}\,)_{\hat{i}}{}^j + \cdots . \tag{6.2}
$$

The pair $\{\,\Phi_i{}^j(\tau)\,,\,\Psi_{\hat{i}}{}^j(\tau)\,\}$ describe a Clifford algebraic superfield. The expansions (6.2) terminate for any given finite value of N since any antisymmetric product with more than N terms vanishes. (i.e., an N-form in N dimensions is a top-form.) Define a supersymmetry transformation according to

$$
\delta_Q(\epsilon)\,\Phi_i{}^j = -i\,\epsilon^I\,(\,L_I\,)_i{}^{\hat{k}}\,\Psi_{\hat{k}}{}^j
$$

$$
\delta_Q(\epsilon)\,\Psi_{\hat{i}}{}^j = \epsilon^I\,(\,R_I\,)_{\hat{i}}{}^k\,\partial_\tau\,\Phi_k{}^j . \tag{6.3}
$$

These rules automatically satisfy (2.1) since by construction L_I and R_I obey (5.1).

One can apply (6.3) to extract the transformation rules level-by level in the expansion (6.2). This requires a careful use of the expressions in (5.1). In the general case, superfield transformation rules (6.3) imply the following rules for the level-expansion

$$
\delta\,\phi^{[p_{\text{even}}]} = -i\,\epsilon^{[I_1}\,\psi^{I_2\cdots I_p]} + (p+1)\,i\,\epsilon_J\,\psi^{I_1\cdots I_p\,J}
$$

$$
\delta\,\psi^{[p_{\text{odd}}]} = -\epsilon^{[I_1}\,\dot{\phi}^{I_2\cdots I_P]} + (p+1)\,\epsilon_J\,\dot{\phi}^{I_1\cdots I_P\,J} . \tag{6.4}
$$

Notice that the first term in $\delta\,\phi^{[p]}$ is a p-form obtained as a wedge-product between the one-form parameter ϵ^I and a fermionic $(p-1)$-form. In the case $p=0$ this term vanishes because $(p-1)<1$, and therefore there is no corresponding fermion.

A word on terminology. In traditional superfields $\mathcal{S}=\mathcal{S}(\,t\,,\,\theta^1\,,...,\,\theta^N\,)$ one refers to the sequence of component fields in terms of "lowest component" to "highest component" where, roughly speaking, the component number corresponds to the associated power of θ^I which appears in a formal Taylor series expansion of $\mathcal{S}$. In Clifford algebra superfields we refer to the analogous sequence using the terms "level-zero" to "level-N". In this case the "level" corresponds to the $\wedge\mathcal{GR}(d_N,N)$ form-degree of the terms in question. For each choice of N there are two distinct Clifford algebra superfields. One has a level-zero boson and one has a level-zero fermion. We refer to the former as a bosonic Clifford algebra superfield and to the latter as a fermionic Clifford algebra superfield. The bosonic Clifford algebra superfield is also called the "base superfield" for the corresponding value of N. In the case of a bosonic

Clifford algebra superfield, the even levels are described by the field $\Phi_i{}^j(\tau)$ and the odd levels are described using the field $\Psi_{\hat{\imath}}{}^j(\tau)$. For instance, the $N = 3$ base superfield has a level-zero boson ϕ, three level-one fermions organized as a vector ψ^I, three level-two bosons organized as a two-form ϕ^{IJ}, and one level-three fermion organized as a three-form ψ^{IJK}. The three form is equivalently described as a one-form in terms of $(*\psi)^I = \varepsilon^{IJKL}\psi_{JKL}$.

7 Root Superfields

Clifford algebraic superfields describe only a restricted class of multiplets. Moreover, for the cases $N \geq 4$ these representations are reducible. This construction complements the superfields described previously using elements of $\mathcal{V}_L \oplus \mathcal{V}_R$ as component fields. Nevertheless, these two sorts of superfields do not yet provide a sufficient basis for a comprehensive representation theory. A big step in that direction is obtained by using the Clifford algebraic superfields as a "base" upon which a variety of operations can be performed so as to obtain a much larger class of representations.

Take a Clifford algebraic superfield (6.2), and write the components as [6]

$$(\phi \, , \, \psi^I \, , \, \phi^{IJ} \, , ...) \;\; = \;\; (\, \partial_\tau^{-a_0} \, \tilde{\phi} \, , \, \partial_\tau^{a_1} \, \tilde{\psi}^I \, , \, \partial_\tau^{-a_2} \, \tilde{\phi}^{IJ} ...) \, , \tag{7.1}$$

etcetera, where $a_i \in \mathbb{Z}$. For the case where all of the a_i are zero, the components $(\tilde{\phi} \, , \, \tilde{\psi}^I \, , \, \tilde{\phi}^{IJ} \, , ...)$ describe the base superfield. However, when at least one of the labels is non-zero, then the structure of the superfield changes in an interesting way. For instance, when one of the bosonic labels is 1, this means that the corresponding component is written as the anti-derivative of a "dual" component. To be more concrete, if $a_2 = 1$ this would imply that $\phi^{IJ}(\tau) = \int^\tau d\tilde{\tau} \, \tilde{\phi}^{IJ}(\tilde{\tau})$ or, equivalently, that $\partial_\tau \phi^{IJ} = \tilde{\phi}^{IJ}$. Note that this describes an automorphic duality transformation. The relationship between the base mutliplet and a generic root multiplet is described in terms of sequences of AD maps.

It is also important to realize that the usual level of a component field in the conventional superspace approach is no longer rigidly linked to the order of the Clifford algebra elements when at least one of the exponents is non-vanishing.

The root superfields utilizing various choices of a_i, in general describe distinct representations of supersymmetry. It is useful to invent a nomenclature to refer

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[6]The convention used here is slightly different than the convention defined in [1]. In this modified convention, the *odd* labels $a_{i=\text{odd}}$ in the base superfield differ by a minus sign as compared to that paper.

to these. Accordingly, we designate the base multiplet, where all of the a_i vanish, using a so-called root label $(0...0)_+$, which includes $N+1$ zeros. The subscript $+$ designates that the zero form is a boson. In the case where the zero form is a fermion, the corresponding root-label is $(0...0)_-$. Starting with the base superfield, another superfield is obtained by dualizing on one of the component levels. For instance, if we started with the $N=3$ base superfield $(0000)_+$ and dualized at level-two, i.e., dualized the two-form ϕ^{IJ}, then we would obtain the superfield $(0010)_+$. Other cases are labelled similarly.

At each value of N the base superfield $(0...0)_+$, plays a special role in the representation theory. It proves helpful to give this multiplet the special symbol $\Omega_{0+}^{(N)}$. Similarly, we denote the Clifford algebraic superfield having a fermionic zero-form, i.e., $(0...0)_-$, as $\Omega_{0-}^{(N)}$.

The numbers a_i in the root superfield label $(a_0, ..., a_N)_\pm$ can take on any integer value. However, the multiplets for which $a_i \in \{\, 0\,,\, 1\,\}$ are of particular interest. We refer to the set of such multiplets as the "root tree". In these cases, the label can be read as a binary number. For instance, the sequence of numbers in the label $(0101)_+$ can be read as $0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 5$. We therefore denote this multiplet using the notation $\Omega_{5+}^{(3)}$. In this way, we can describe a useful class of multiplets using the concise names $\Omega_{\mu\pm}^{(N)}$, where μ are integers such that $0 \leq \mu \leq (\, 2^{N+1} - 1\,)$. As it turns out, there is in general much redundancy in these names. For instance, for each choice of N, the multiplets in the root tree having root label $(\, 1\,,\, a_1\,, ...,\, a_N\,)_\pm$ are the same as the multiplets $(\, 0\,,\, 1 - a_1\,, ...,\, 1 - a_N\,)_\pm$. Thus, without loss of generality we consider $0 \leq \mu \leq (\, 2^N - 1\,)$.

8 Multiplet Adinkras

In this section we define a powerful diagrammatic technique which usefully encodes many aspects of supersymmetry multiplets. According to this scheme, each multiplet has a corresponding distinctive symbolic form, which we refer to as an adinkra symbol, or an adinkra for short. An adinkra uses white circles to represent bosons and black circles to represent fermions. In either case the circles are called nodes. The nodes are interconnected using oriented line segments, referred to as arrows. The arrows are directed from nodes representing lower component fields toward nodes representing higher component fields.

8.1 $N = 1$ Adinkras

Each of the two irreducible $N = 1$ multiplets include off-shell one bosonic and one fermionic degree of freedom. Accordingly, the adinkras for these multiplets include one white node, to represent the boson, and one black node, to represent the fermion. In the case of the scalar multiplet, the boson is the lower component and the fermion is the higher component. Accordingly, the adinkra for the scalar multiplet is

Since the arrow points toward the black node, it is clear that the fermion is the higher component in this multiplet. We can use the adinkra as a method for identifying this multiplet. We recall, however, that the scalar multiplet can also be described with a root label, as $(00)_+$, or using Omega notation, as $\Omega_{0+}^{(1)}$. Each of these three schemes has advantages and disadvantages. In the balance of this paper we demonstrate how the adinkra is useful for organizing the assembly of $N = 1$ multiplets into higher-N multiplets, for identifying irreducible multiplets, and for describing gauge invariance. We will use root labels or Omega notation in cases where these choices are advantageous, however, since the three notational schemes usefully complement each other.

In the case of the spinor multiplet the fermion is the lower component and the boson is the higher component. Accordingly, the adinkra for the spinor multiplet is

Since the arrow points toward the white node, it is clear that the boson is the higher component in this multiplet. The spinor multiplet can also be described with a root label, as $(00)_-$, or using Omega notation, as $\Omega_{0-}^{(1)}$.

The adinkras symbolically encode the supersymmetry transformation rules for the corresponding multiplets. This is seen easily in the $N = 1$ case by comparing the adinkras shown above with the transformation rules given in (2.2). The supersymmetry transformation rule for a generic component field $f(\tau)$ corresponding to an adinkra node is given in this case by

$$\delta_Q(\epsilon)\, f = \pm i^b\, \epsilon\, \partial_\tau^\lambda f\,, \tag{8.1}$$

where $b = 1$ for bosons and $b = 0$ for fermions, and $\lambda = 1$ for lower components and $\lambda = 0$ for higher components. The ambiguous sign appearing in this rule must be chosen identically at each of the two nodes. The choice of which sign is irrelevant in the $N = 1$ case, since this can be flipped by redefining either node with a multiplicative minus sign [7]. We refer to this sign choice as the "arrow parity". The concept of arrow parity becomes important when combining $N = 1$ multiplets to form higher-N multiplets, as we explain below. The reader is encouraged to derive the transformation rules (2.2) from the two $N = 1$ adinkras presented above. This is a simple exercise which illustrates only a part of the hidden meaning in these symbols. The useful mnemonic is that each boson receives a factor of i in its transformation rule, and higher components appear differentiated. This rule generalizes to generate the transformation rule corresponding to any arrow in any of the adinkras in the root tree for any value of N, but requires modification for adinkras not in the root tree.

An adinkra symbol does not have an intrinsic orientation; either of the adinkras shown above can be rotated arbitrarily in the plane of the page. For certain purposes, it is useful to draw the symbol in a particular manner, however. For instance, it is conventional to present transformation rules starting with the lowest component at the top of a list, and work toward the highest component at the bottom of a list. The adinkra most faithfully represents this structure if all arrows point downward. This was the choice made in the case of the scalar adinkra, shown above, but not in the case of the spinor adinkra. The choice made in the case of the spinor adinkra serves a different purposes, as will become clear. Since arrows typically point from lower components to higher components we refer to a node as being "higher" than an adjacent node if an arrow points from the former node toward the latter node. For multiplets in the root tree all nodes conform to an unambiguous hierarchy such that each node is either higher, lower, or at the same height as each of the other nodes. There are interesting other adinkras, not in the root tree, for which there is not an unambiguous hierarchy.

Recall that the scalar multiplet can be mapped into the spinor multiplet using an AD map. The effect of this map is to exchange the roles of which of the two adjacent nodes is higher or lower. Accordingly, this map can be visualized as a reversal of the "sense" on the arrow connecting the two adjacent nodes in the adinkra,

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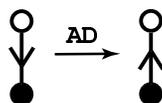
[7]We refer to nodes and to the corresponding component fields as if they were the same entity. Thus, by scaling a node we mean that we are scaling the corresponding field.

We can interpret the boson in the scalar multiplet as the lowest level in the Clifford algebra superfield $(00)_+$. The boson in the spinor multiplet can be interpreted as the lowest level in the root superfield $(01)_+$. Thus, if the adinkra is oriented in such a way that the superfield levels are manifest, then by representing the AD map as an operation which reverses the sense of an arrow, but which leaves the position of the nodes unchanged, we faithfully preserve the manifestation of levels on the structure of the adinkra.

When we implement the map described above, $(00)_+ \rightarrow (01)_+$ we have replaced the level-one fermion in $(00)_+$ with a dual component. Accordingly, we say that we have "dualized" at level-one. This is readily visualized not only on the adinkra, but also on the root label, since level-one corresponds to the second index in the root label. The rule for implementing AD maps on adinkra symbols is that dualizing at a certain level corresponds to flipping all arrows with connect to nodes at that level.

Suppose we implement a different AD map, this time by dualizing at level-zero. This corresponds to $(00)_+ \rightarrow (10)_+$, since level-zero corresponds to the first index in the root label. Using the rules described above, we represent this by reversing the sense of every arrow connecting to the top node in the $(00)_+$ adinkra. This produces the same diagram obtained by our previous duality operation. In other words, the adinkra for $(10)_+$ is the same as the adinkra for $(01)_+$. Thus, the $N = 1$ spinor multiplet is described by the equivalent labels $(01)_+ \cong (10)_+$.

The $N = 1$ spinor multiplet is also described by the label $(00)_-$, since it corresponds to the $N = 1$ Clifford algebra superfield having a level-zero fermion. Using this label, an AD map could be performed at level-one as $(00)_- \rightarrow (01)_-$. This is implemented by reversing the sense of every arrow which connects to the level-one node in the spinor multiplet adinkra. Each of the AD maps described so far merely toggle between the two possible $N = 1$ adinkras. We discover in this way a nexus of congruencies in the root labels. Specifically, $(00)_+ \cong (01)_- \cong (10)_-$ and $(00)_- \cong (01)_+ \cong (10)_+$. In terms of Omega notation, this result corresponds to $\Omega_{0+}^{(1)} \cong \Omega_{1-}^{(1)} \cong \Omega_{2-}^{(1)}$ and $\Omega_{0-}^{(1)} \cong \Omega_{1+}^{(1)} \cong \Omega_{2+}^{(1)}$. In the $N = 1$ case AD maps comprise $\mathbb{Z}_2$ generators which link the two congruency classes.

There is another useful $\mathbb{Z}_2$ map which is distinct from the AD maps described so far. This second map is described by replacing every bosonic node in a given adinkra with a fermionic node, and vice-versa. This operation was introduced in [7], where it was deemed a Klein flip. In the case of $N = 1$ supersymmetry a Klein flip has the same effect as an AD map, since it toggles between the two adinkras. The circumstance that AD maps and Klein flips generate indistinguishable automorphisms is special to

the case $N = 1$ where it is a consequence of the relative simplicity of the space of irreducible representations.

9 $N = 2$ Adinkras

Consider the $N = 2$ scalar multiplet described by (4.2). To be specific, chose the $\mathcal{GR}(2,2)$ matrices according to $L_1 = R_1 = i\,\sigma_2$ and $L_2 = -R_2 = -\mathbb{I}_2$, where $\mathbb{I}_d$ is the $d \times d$ unit matrix. This describes the unique representation of $\mathcal{GR}(2,2)$. It is easy to translate the corresponding transformation rules into an adinkra symbol using the rules described above. The result is

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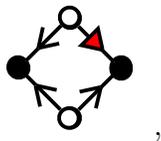
where we have distinguished one of the arrows for a reason to be explained shortly. But first, we explain the general structure of this adinkra. Each pair of parallel arrows corresponds to one of the two supersymmetry transformations. For the sake of concreteness, lets say that the red arrow and the arrow opposite the red arrow correspond to the first supersymmetry, described by parameter ϵ^1, and that the remaining two arrows correspond to the second supersymmetry, parameterized by ϵ^2.

The reader is encouraged to use (4.2) along with the representation of $\mathcal{GR}(2,2)$ given above, to verify the rule (8.1) node-by-node and arrow by arrow. To do this, let the top node represent ϕ_1 and the bottom node represent ϕ_2, and let the left node represent $\psi_{\hat{1}}$ and the right node represent $\psi_{\hat{2}}$. This exercise will expose the special characteristic which distinguishes the red arrow in this diagram. Namely, this arrow corresponds to a choice of minus sign in (8.1), whereas the remaining three arrows correspond to a choice of plus sign in this rule. This indicates a topological characteristic required of any adinkra symbol for the cases $N \geq 2$, as explained presently. Each of these symbols has closed circuits which can be traced on the diagram by following arrows from node to node. A consequence of the minus signs in the first two equations in (5.1) is that the sum of the arrow parities associated with any four-node closed circuit must be negative [8].

It is possible, of course, to redefine any component field by use of a multiplicative minus sign. We refer to this benign operation by saying that we have flipped the

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[8] Note that there is no correlation between the parity of an arrow and its orientation.

sign of a node. Notice that by flipping a sign on either of the nodes adjacent to the negative-parity arrow, the position of the negative parity arrow shifts around the diagram. Another possibility is to flip the sign on one of the nodes not adjacent to the negative parity arrow. The effect of this is to grow two more negative parity arrows. Note that in the case of the square diagram flipping node signs necessarily changes the parity of exactly two arrows. In this way the sum rule is preserved under such operations, although the parity of any given arrow can be flipped by field redefinitions.

The reader might imagine that keeping proper track of arrow parities could become a complicated business in higher-N diagrams. Fortunately, there is a simple algorithm which handily takes care of this for us in many, if not all, circumstances. This algorithm relies on the root superfields described in section 7, each of which is derived from the base superfields by AD maps and Klein flips. The arrow parities for the base superfields are consistently dictated by the transformation rules given in (6.4). As a result, one can draw the adinkra symbol for a base superfield without specifying the arrow parities, knowing that these can be chosen in a consistent manner. It is then possible to derive a variety of related multiplets by implementing AD maps (by flipping arrows) and Klein flips (by flipping node colors), again without regard for arrow parity, since consistency is ensured by the fact that the base adinkra is consistent by construction.

It is a noteworthy fact that the $N = 2$ scalar multiplet described above is, in fact, the $N = 2$ base multiplet $\Omega_{0+}^{(2)}$. This can be verified by determining the transformation rules for $\Omega_{0+}^{(2)}$ using (6.4), and then translating these into an adinkra symbol. The reader is encouraged to do this.

We are now in a position to describe the $N = 2$ root tree using adinkra symbols. Consider the following four $N = 2$ adinkras,
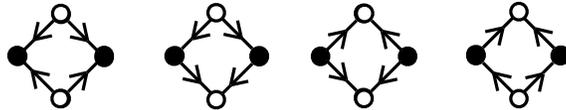
The first of these is the base adinkra $(000)_+$ which we have described at length already. By convention, we have suppressed any special markers indicating arrow parity, since as explained above these are not necessary. We have oriented this adinkra such that the top node corresponds to level-zero, the middle nodes correspond to level-one, and the bottom node corresponds to level-two. The second of the adinkras shown here is obtained from the first by dualizing at level-two. This is clear because the relationship between the second adinkra and the first is that both arrows which connect with the

level-two node have been flipped. This second multiplet has root label $(001)_+$ and Omega designation $\Omega_{1+}^{(2)}$. The third adinkra is obtained from the first by flipping all arrows which connect to level-one nodes. (In this case this describes all arrows!) Thus the third multiplet has root label $(010)_+$ and Omega designation $\Omega_{2+}^{(2)}$. The fourth adinkra is obtained from the first by flipping both arrows which connect with the level-zero node. Thus, this multiplet has root label $(100)_+$. This fourth adinkra is also obtained by flipping all arrows connecting to the level-one nodes and then flipping both arrows connecting to the level-two nodes. According to this second interpretation the fourth adinkra has the equivalent root label $(011)+$ and Omega designation $\Omega_{3+}^{(2)}$. The three distinct adinkras $\Omega_{\mu+}^{(2)}$ where $\mu = 0, 1, 2$, form a sequence, which we refer to as the "base sequence" for $N = 2$.

The $N = 2$ adinkras presented so far describe all of the adinkras which can be obtained from the base adinkra by AD maps. Notice that the $(001)_+$ adinkra is homologous to the $(100)_+ \cong (011)_+$ adinkra. This is seen by rotating either of these by 180 degrees. Thus, $(001)_+ \cong (100)_+$, or equivalently $\Omega_{3+}^{(2)} \cong \Omega_1^{(2)}$. Thus, the number of multiplets in the AD orbit connected to the $N = 2$ base multiplets is three, not four.

There is still another multiplet in the $N = 2$ root tree yet to be described. To locate this missing multiplet, consider that multiplet obtained from the base mutliplet by implementing a Klein flip, namely $(000)_-$. Consider as well the set of multiplets connected with this one via AD maps. This set is described by the following four adinkra symbols,
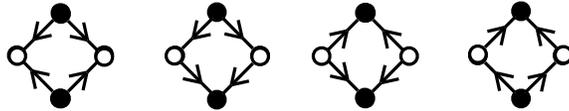
The first of these is the image of the base adinkra under a Klein flip, i.e., $(000)_-$. The other three adinkras shown here are obtained from this one by dualizing at levels two, one and zero, respectively. Accordingly, these describe the respective multiplets $(001)_-$, $(010)_-$ and $(100)_- \cong (011)_-$ or, using Omega notation, $\Omega_{1-}^{(2)}$, $\Omega_{2-}(2)$ and $\Omega_{3-}^{(2)}$, again respectively. This sequence of four adinkras describes the respective images under Klein flips of the base sequence shown previously. This is easily verified by flipping the color of all nodes and then comparing these two sequences. By rotating the $\Omega_{3-}^{(2)}$ adinkra by 180 degrees, we observe that this is homologous to the $\Omega_{1-}^{(2)}$ adinkra. The three distinct adinkras $\Omega_{\mu-}^{(2)}$ where $\mu = 0, 1, 2$, form a sequence, which we refer to as the "mirror sequence" for $N = 2$.

We now see more congruencies, this time showing equivalences between elements of the base sequence with elements in the mirror sequence. For instance, by rotating the $\Omega_{0-}^{(2)}$ adinkra by 90 degrees, we see that this is identical to the $\Omega_{2+}^{(2)}$ adinkra. Also, by rotating the $\Omega_{2-}^{(2)}$ adinkra by 90 degrees we see that this is identical to the $\Omega_{0+}^{(2)}$ adinkra. Therefore, the only adinkra which appears in the mirror sequence which is distinct from all of those in the base sequence is $\Omega_{1-}^{(2)}$. In this way we determine that the $N = 2$ root tree has four elements, which can be described as $\Omega_{0\pm}^{(2)}$ and $\Omega_{1\pm}^{(2)}$.

9.1 Adinkra Folding

We have seen that the irreducible $N = 1$ adinkra symbols each comprise two nodes connected by an oriented line segment. Thus, these symbols span only one linear dimension. By way of contrast, the irreducible $N = 2$ adinkra symbols comprise four nodes configured at the corners of a square, which has oriented line segments as edges. Thus, the $N = 2$ adinkras span two dimensions. The reason for this is that the two supersymmetries are represented by orthogonal arrows. Following this logic, the $N = 3$ adinkras span three dimensions, so that the three supersymmetries can be represented using three mutually-orthogonal sets of arrows. As N increases this leads to complicated symbols which would be difficult to render on a page. However, there is a useful operation one can perform on adinkras, which allows the drawing of many of these for any N as a linear chain, yet retains the full symbolic power. In this subsection we describe this process for the case $N = 2$, although the technique generalizes to higher N.

We can squash or fold any adinkra by moving bosonic nodes onto other bosonic nodes and at the same time moving fermionic nodes onto other fermionic nodes, while simultaneously maintaining all arrow-node connections. This can be done provided that all arrows land on identically oriented arrows. For instance, consider the $\Omega_{1+}^{(2)}$ adinkra shown above. In this case, we can pinch the two fermionic nodes together, as follows,
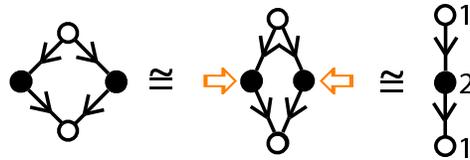
At the end of the process, the two fermionic nodes are coincident. We indicate the multiplicity of this compound node by placing a numeral 2 next to the node. Nodes

representing more degrees of freedom occur in more complicated adinkra symbols. The node multiplicity is indicated by a numeral placed adjacent to the node.

In many cases, the arrows in an adinkra symbol are structured in such a way that permits additional folds after the first one. Consider, for instance the $N = 2$ base adinkra, which can be folded as follows,
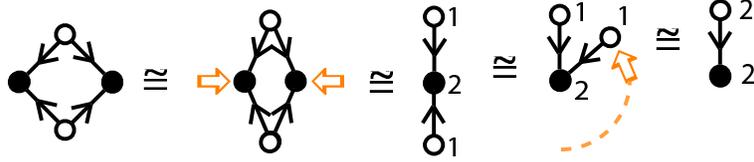
In this case we begin by pinching the two fermionic nodes together in a manner identical to the operation performed above on the $\Omega^{(2)}_{1+}$ adinkra. In this case, since all the arrows continue to point to the multiplicity-two fermionic node, it is possible to swivel the bottom bosonic node, using the compound fermionic node as a pivot, until it coincides with the top bosonic node. In this way we obtain a folded form which involves two multiplicity-two nodes, one bosonic and one fermionic, connected by one arrow which now represents both supersymmetries.

By using similar folding operations, all of the elements of the root tree for any value of N can be arranged into a linear chain. Many other adinkras, which are not elements of the root tree can not be folded into a linear chain; these describe an interesting class of multiplets which is described in the following section. Each distinct adinkra has a fully-folded form which is distinct from the fully-folded forms of all other distinct adinkras. It is possible to identify each distinct supersymmetric multiplet with a unique fully-folded adinkra symbol. The folded adinkras can be unfolded, using certain rules, in such a way as to recover the fully unfolded adinkra. As we explain below, it is often useful to start with a fully-folded adinkra symbol, and then only partially unfold this before implementing duality maps, by making arrow reversals. This can then be re-folded to obtain a new adinkra.

10   Escheric Multiplets

In this section we describe some surprising unanticipated aspects of supersymmetry representations which become evident when this subject is structured in terms of adinkra symbols. We present these observations at this point, immediately following our description of the basic $N = 2$ adinkras, because these aspects are most clearly

illustrated in the context of $N = 2$ supersymmetry. We continue the main thrust of the paper, by generalizing our technology to the cases $N = 3$ and $N = 4$, in the sections which follow this one.

By including AD maps and Klein flips together, we are able to effectively realize dualities on the $N = 2$ base adinkras node-by-node rather than level-by-level. To illustrate this, start with the $N = 2$ base adinkra, perform a Klein flip, then dualize at level-two in the resulting adinkra, then rotate the adinkra by 90 degrees,
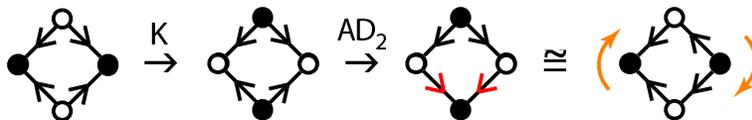
The result of this sequence of operations is the same as if we dualized on only one of the two level-one fermion nodes in the base adrinkra. In other words, this operation is equivalent to
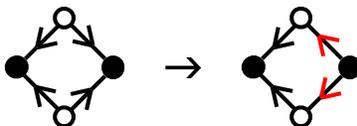
This begs the question as to whether we can realize dualities node-by-node rather than level-by-level as a general rule. The answer is that, generally, such transformations cannot be implemented by a combination of Klein flips and AD maps. The reason for this is connected to the fact that the AD maps, as described above, act in a strictly level-specific manner on the root superfields. The case of the $N = 2$ base adinkra provides an exception, as we have seen. Is it nevertheless possible to implement dualities node-by-node on any given adinkra? Does this supply us with new representations of supersymmetry? The answer to both questions is, interestingly, yes. These operations generally produce new multiplets which lie outside of the root tree and which have noteworthy nontrivial topological features. It is also possible in this way to obtain multiplets which represent centrally-extended superalgebras.

The simplest example of this phenomenon is described by starting with the $\Omega_{2+}^{(2)}$ adinkra and dualizing at one of the two level-one fermionic nodes. As it turns out, there are two rather different senses to interpret "dualization" in this context. The first sense is to simply reverse the sense of each arrow which connects to the node being dualized. As we will see, in the current context this process produces a rather different sort of multiplet, one which does not strictly represent the superalgebra

(4.1), but rather represents a centrally-extended version of this algebra. As a result, this new multiplet cannot be described in terms of the basic root superfields described above. The second sense in which dualization can be interpreted is more properly aligned with the duality maps described so far. In this second sense we identify the designated fermionic node with the proper-time derivative of a "dual" node. In this second sense, we obtain a multiplet which can be described by a root superfield, and which does represent the superalgebra (4.1). But this multiplet lies outside the root tree. For reasons made more clear in the Appendix this multiplet requires a slight modification to the diagrammatics introduced to this point. In cases where AD maps are implemented level-wise, the two senses of dualization described above coincide. Otherwise, as we have just explained, theses senses differ, and each sense maps multiplets in the root tree into multiplets outside the root tree.

First, lets consider dualization in the first sense. By reversing both arrows which connect to only one of the two level-one fermion nodes in the $\Omega_{2+}^{(2)}$ we obtain the following new adinkra,
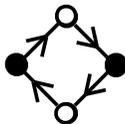
This particular adinkra symbol has several noteworthy features. First of all, it is impossible to fold this adinkra into a linear form. This is because of a topological obstruction which relates in an interesting way to the corresponding transformation rules. As explained above, a given node is designated as "lower" than another node if the second node can be reached from the first by using the arrows to define a flow pattern. Nodes which are downstream in such a flow describe higher components. For the interesting multiplet shown above, each node is at the same time both upstream and downstream of every other node; there is no highest component and there is no lowest component. We shall refer to multiplets with this feature as escheric multiplets, owing to the similarity with patterns found in many drawings of M. C. Escher. Another, rather surprising feature is manifested by writing down the corresponding transformation rules using the procedure described above. This is done in the Appendix. It turns out that this multiplet does not strictly represent the $N = 2$ superalgebra (4.1). Instead, this represents a centrally-extended version of this superalgebra. At the time of this writing, the implications of this are not perfectly clear. But we find this intriguing.

Next, lets consider dualization in the second sense. By writing one of the two level-one fermions in the $\Omega_{2+}^{(2)}$ adinkra as the proper-time derivative of a dual fermion,

we obtain a new multiplet which does properly represent the $N = 2$ superalgebra without a central charge. However, in this case the transformation rules for one of the fermions include the antiderivative of one of the boson fields. The details are explained in the Appendix, where it is also shown that this multiplet includes as a sub-multiplet the $N = 1$ root multiplet $(2, 0)_+ \cong (0, 2)_+ \cong (0, -1)_- \cong (-1, 0)_-$. Since these labels include integers which are neither 0 nor 1, this multiplet lies outside the $N = 1$ root tree. Accordingly, this is not properly described by the adinkra symbols defined to this point. (A relevant addendum to the notation introduced above is also included in the Appendix.) We refer to this sort of multiplet as a "type II escheric multiplet" to distinguish these from the central charge escherics described previously. The appearance of the antiderivatives in the transformation rules is another circumstance which may have interesting relevance to physics, especially in cases where the corresponding field describes the coordinate on a compact dimension.

In this paper we are concerned principally with the elements of the root tree. As a result we will not describe escheric multiplets any further in this main text. More details are included in the Appendix. We intend to study these constructions further in ongoing work, and hope to have more to say on this topic in the future.

11 $N = 3$ Adinkras

Consider the $N = 3$ scalar multiplet described by (4.2). To be specific, choice the $\mathcal{GR}(4, 3)$ matrices according to $L_1 = R_1 = i\,\sigma_1 \oplus \sigma_2$, $L_2 = R_2 = i\,\sigma_2 \otimes \mathbb{I}_2$ and $L_3 = R_3 = -i\,\sigma_3 \otimes \sigma_2$. It is easy to translate the corresponding transformation rules into an adinkra symbol using the rules described above. The result is
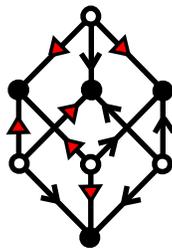
In this adinkra, the four bosonic and the four fermionic nodes are situated at the corners of a cube. Each of three quadruplets of parallel arrows corresponds to a different supersymmetry.

In rendering the adinkra shown above we have distinguished the negative-parity arrows by giving these red color. We have done this in order to make a couple of

basic points. First of all, it is easy to verify the sum rule, described above, which says that the sum of the four arrow parities associated with any square sub-adinkra, must be odd. This rule is easily verified on this diagram by tracing around each of the six faces of the cube, counting arrow parities in the process. Next, recall that the arrow parities of every arrow connected to a given node flips when the node has its sign flipped. In this way, the position of the negative parity arrows can be shifted around the adinkra symbol without changing the representation. In the case of the cubic adinkra shown here, each time a node has its sign flipped, three arrows have their parity flipped. This process flips either exactly zero or exactly two arrows in each subset of four arrows forming the edges of each face. Since zero and two are even numbers this proves that the sum rule is maintained when any node has its sign flipped.

Starting with the adinkra shown above, it is possible to cycle through a sequence of node flips, that cycles through all possible distributions of negative-parity arrows which satisfy the sum rule. We will not describe a complete proof of this statement in this paper, however. This shows that the $\mathcal{GR}(4,3)$ representation given above is unique.

Consider next the $N = 3$ base multiplet. This has transformation rules given by (6.4). If we translate these into an adinkra symbol we find that this symbol is identical to the $N = 3$ scalar adinkra shown above. By following the folding rules described above, it is possible to reduce the $N = 3$ base adinkra into a linear form. This diagram can be folded as follows,
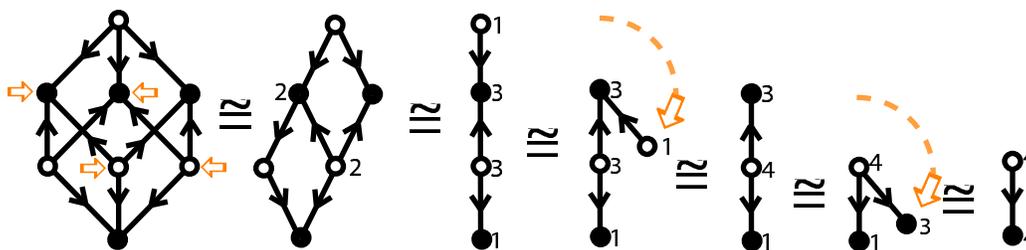
In this sequence, we first pinch together two of the bosonic nodes and two of the fermionic nodes, as shown, thereby collapsing two opposite square faces into lines. This step reduces the figure to two dimensions. Next we pinch the resultant multiplicity-two boson node together with another bosonic node, forming one multiplicity-three bosonic node. At the same time we do a similar thing to form a multiplicity-three fermionic node. This reduces the adinkra into a linear form. Two additional folds then transform the adinkra into its final form, given by two multiplicity-four nodes,

one bosonic and one fermionic, connected by an arrow representing all three super-symmetries.

The $N = 3$ base adinkra has root label $(0000)_+$ and Omega designation $\Omega_{0+}^{(3)}$. The level-zero bosonic node corresponds to the topmost node appearing in the two-dimensional projection of the fully-unfolded form of adinkra shown above. Successive levels in the root superfield correspond to the sequence of horizontal node groupings which appear in this projection. We can form distinct multiplets starting with the $N = 3$ base adinkra, by performing AD maps and Klein flips. For example, we dualize at level-three by flipping all arrows which connect to the level-three boson (the bottommost node in the above diagram.) Doing this and then folding the resultant adinkra, we observe the following,
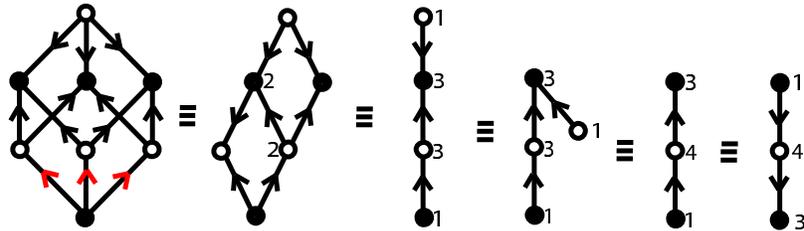
In the final step we have rotated the adinkra by 180 degrees, so that all arrows point downward. In this case, we see that the arrow structure precludes the analog of the final fold made previously in the case of the $(0000)_+$ adinkra.

As another example, start with the $(0001)_+$ adinkra and dualize at level-two. This generates the map $(0001)_+ \to (0011)_+$. This is implemented by flipping all arrows which connect to the level-two nodes in the fully unfolded $(0001)_+$ adinkra. In this way we obtain
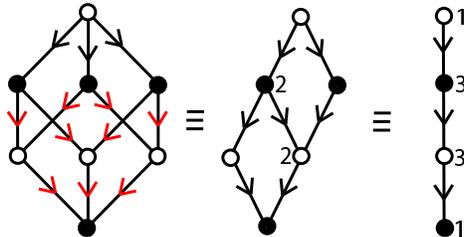
Here fewer folds are permitted by the arrow structure than in the previous case, so that the fully-folded $(0001)_+$ adinkra has four nodes, rather than two. The maximal number of nodes in a fully folded adinkra is $N + 1$. We denote the fully folded adinkra which has $N + 1$ nodes as the "top adinkra" for that value of N.

As a final example, start with the $(0011)_+$ adinkra and dualize at level-zero. This generates the map $(0011)_+ \to (1011)_+ \cong (0100)_+$. This is implemented by reversing all arrows connecting to the level-one nodes in the $(0011)_+$ adinkra. In this way we obtain
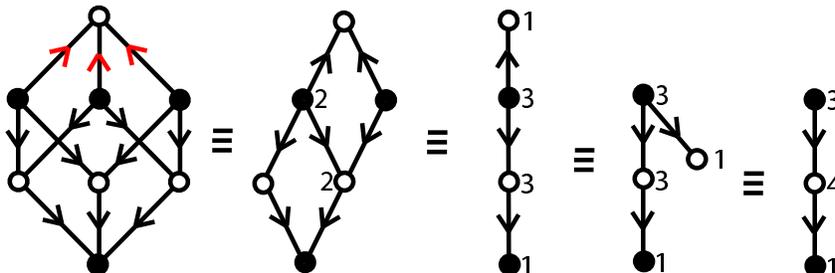
In this case we see that the fully-folded adinkra has three compound nodes, having the multiplicities shown in the final diagram.

As described previously, every adinkra in the root tree for any value of N can be folded into a linear chain having $N+1$ nodes. Most adinkras can be folded further, so that in fully-folded form these exhibit fewer than $N+1$ compound nodes. The number of compound nodes in the fully-folded form corresponds to the number of distinct component heights. On the other hand, the partially-unfolded form describing a chain with $N+1$ nodes is more useful for implementing AD maps. This is because in this form the adinkra nodes sequentially correspond to root superfield levels. Since AD maps are implemented in a level-specific manner, these can be implemented on this form by reversing the compound arrows which connect to the nodes corresponding to desired levels. To implement AD maps on elements of the root tree it is not necessary to unfold the diagram into more than one dimension.

We define the "depth" of an automorphism as the number of dimensions that an adinkra has to be unfolded into before the particular automorphism can be implemented. Thus, the AD maps which we have described above comprise depth-zero automorphisms on the space of superalgebra representations. Arrow reversals and Klein flips at each depth greater than zero form separate abelian groups, since each of these operation generates its own $\mathbb{Z}_2$ subgroup. We define the "rank" of a supermultiplet as one less than the minimal number of dimensions spanned by the fully folded adinkra. In this way the root tree comprises depth-zero multiplets. The escheric multiplets described above correspond to multiplets having depth greater than zero.

12 The $N \leq 3$ Root Trees and Auxiliary Fields

A field is typically deemed "auxiliary" if it describes no dynamical (on-shell) degrees of freedom. However, it is possible to make an equivalent non-dynamical definition using the flow pattern generated by the arrows in adinkra symbols. According to this definition, auxiliary bosons are defined as bosonic nodes which appear as flow sinks, nodes to which all associated arrows point toward. Auxiliary fermions are defined as fermionic nodes which appear as flow sources, nodes from which all associated arrows point away. For all supersymmetric actions with minimal kinetic derivatives, fields deemed auxiliary from the dynamics-free point of view are also auxiliary from the usual dynamical definition. As a notational convention, we sometimes place a box around auxiliary nodes in adinkra symbols.

The 14 adinkras which describe the $N \leq 3$ root trees are shown in Figures 2, 3 and 4. These tables comprehensively exhibit the off-shell state counting for each of the minimal rank-zero multiplets, and also clearly indicate the interconnections between these generated by AD maps and Klein flips. For each choice of N, the adinkras are displayed in cells which include numbers describing the Omega designation for that multiplet. For instance, the $N = 3$ adinkra labelled 4+ corresponds to $\Omega_{4+}^{(3)}$. Extra numbers in any cell correspond to a notational redundancies, such as $\Omega_{7+}^{(3)} \cong \Omega_{4-}^{(3)}$. The root label for any of these multiplets are readily obtained by writing the decimal number in the Omega notation as the binary equivalent. Accordingly, the root label for $\Omega_{4+}^{(3)}$ is $(0100)_+$. This multiplet is obtained from the base multiplet $\Omega_{0+}^{(3)}$ by dualizing on the level-one nodes. The reader is encouraged to verify the tables using the techniques described previously.

Figures 2, 3 and 4 clearly exhibit the $\mathbb{Z}_2$ representation generated by Klein flips; adinkras on the right sides of each tabulation are obtained from those on the left by performing this operation. The correspondence is easy to read, since the Klein flip manifests by flipping the sign on the Omega label. The field multiplicity of each multiplet can be read off of the adinkras. Bosons correspond to white nodes and fermions correspond to black nodes. Boxed nodes correspond to auxiliary fields. For instance, the multiplet $\Omega_{4+}^{(3)}$ is seen to have off-shell fields consisting of three physical bosons and one physical fermion and to have one auxiliary bosons and three auxiliary fermions.
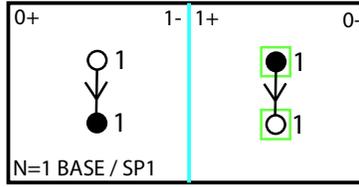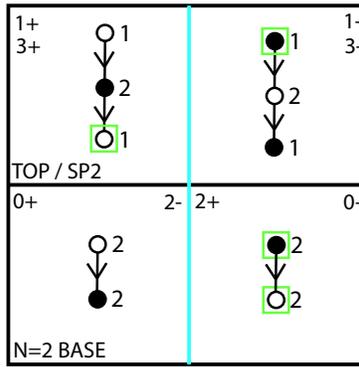
Figure 2: The root-tree for the case $N = 1$.
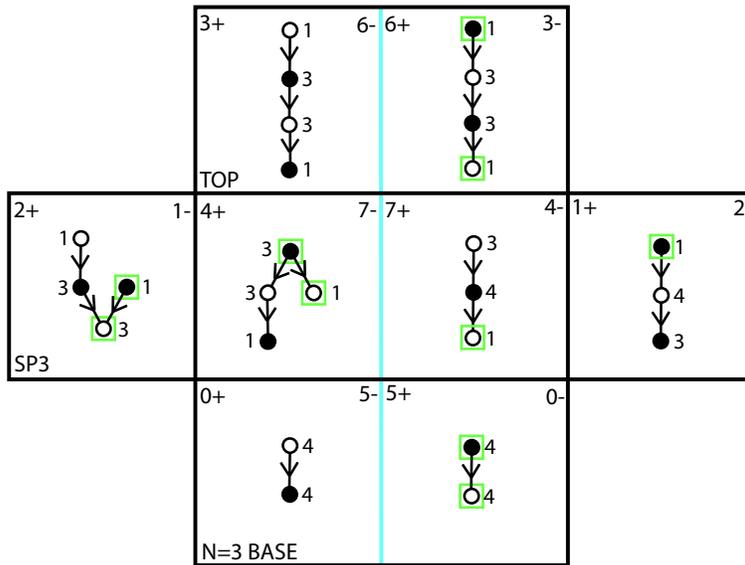

Figure 3: The root-tree for the case $N = 2$.

Figure 4: The root-tree for the case $N = 3$.

13 $N = 4$ Adinkras

New structures appear at $N = 4$ which are absent in the cases $N \leq 3$. The reason for this is that, in contrast to the cases $N \leq 3$, the base multiplets $\Omega_{0+}^{(4)}$, and all representations obtained from this by AD maps and Klein flips, describe reducible representations. For instance, the minimal $N = 4$ multiplets have 4+4 off-shell degrees of freedom, as shown in Table 1. However, each element of the $N = 4$ root tree has 8+8 off-shell degrees of freedom, or twice the minimum. In this section we explain two methods for reducing such multiplets using the structure of adinkra symbols as a conceptual guide. The first method uses consistent node identifications to describe the embedding of irreducible multiplets inside the root space. The second method is to identify irreducible sub-adinkras describing gauge degrees-of-freedom. These methods prove sufficient for describing all known $N = 4$ irreducible multiplets.

The 18 adinkras which comprise the $N = 4$ root tree are shown in Figure 5, which is structured in the same manner as Figures 2, 3 and 4. The Omega designation for each multiplet is clearly indicated, including notational redundancies. Adinkras on the right side of Figure 5 are obtained from those on the left by making a Klein flip.

13.1 Irreducible $N = 4$ Multiplets

A class of irreducible multiplets is described by the scalar multiplets, with transformation rules given in (4.2). These are determined by choosing a representation of $\mathcal{GR}(d_N, N)$. For the case $\mathcal{GR}(4, 4)$, we can make the choice $L_1 = R_1 = i\,\sigma_1 \oplus \sigma_2$, $L_2 = R_2 = i\,\sigma_2 \otimes \mathbb{I}_2$, $L_3 = R_3 = -i\,\sigma_3 \otimes \sigma_2$ and $L_4 = -R_4 = \mathbb{I}_2 \otimes \mathbb{I}_2$. It is possible to translate the transformation rules into an adinkra symbol, but there are extra subtleties not encountered in the cases $N \leq 3$.

The first subtlety is relatively simple. Since each adinkra node connects with N-mutually-orthogonal arrows, one for each supersymmetry, it follows that the fully-unfolded form spans N-dimensions. This makes the unfolded form relatively awkward to render on a page. This problem can be overcome, at least for small values of N, by making small compromises with angles and with parallel lines, or it can overcome quite satisfactorily by folding the diagram down to one or two dimensions, if possible.

The second subtlety has to do with combinatorics. For any value of N the root multiplets have a total of 2^N nodes, just the right number to sit on the corners of an N-dimensional hypercube. The irreducible multiplets have fewer than 2^N nodes, so it is not straightforward to connect the nodes with an N-dimensional orthogonal lattice.
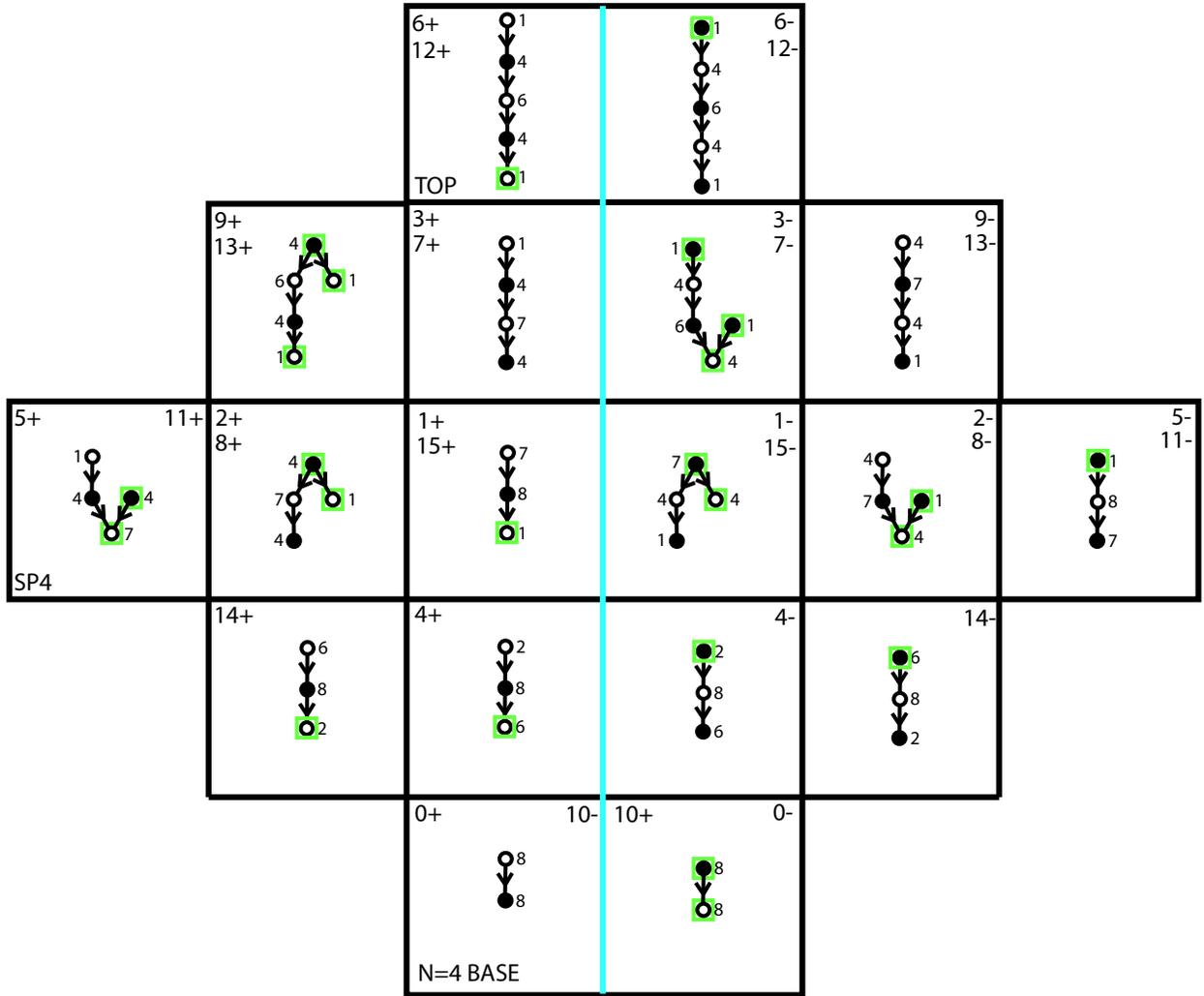
Figure 5: The $N = 4$ root-tree.

Many irreducible adinkras permit an embedding into an N-dimensional orthogonal lattice by including multiple copies of the original adinkra into the lattice. Since there are 16 corners to a tesseract, and 8 total nodes in a scalar adinkra, it is conceivable that a double-copy of the scalar adinkra could fit properly into the tesseract. In fact, this works perfectly well, as can be seen by the following diagram,


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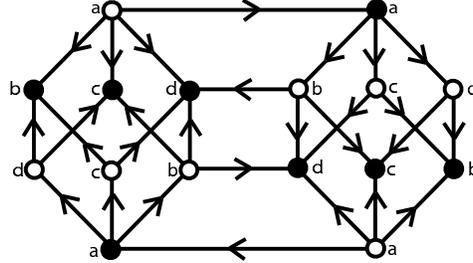
Here we have designated each node with a distinctive label. The cube on the right is an upside-down copy of the cube on the left. Thus the bosonic node a on the top of the left side is the same as the bosonic node a on the bottom of the right side. Each cube describes a representation of an $N = 3$ subalgebra. The horizontal arrows describe the fourth supersymmetry. We have suppressed arrows pointing from the white c node to the black c node and from the white d node to the black d node, so as not to confuse the diagram. This represents an accurate depiction of the transformation rules corresponding to the $N = 4$ scalar multiplet described above. The adinkra diagram corresponding to the $N = 4$ base multiplet is the same as the double-box diagram shown above, except with all of the identifications removed. Thus, this method shows a way to embed the scalar multiplet into the $N = 4$ root space.

We can simplify the presentment of the scalar adinkra by folding the $N = 3$ subdiagram in the manner described in section 11. In this way, the diagram takes the simpler form
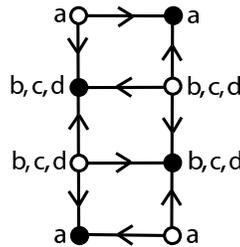

In the folded form, the pairwise node identifications remain indicated by labels. Another simplifying convention is to divide the two equivalent parts of this diagram using a "mirror plane", and to redraw as follows,
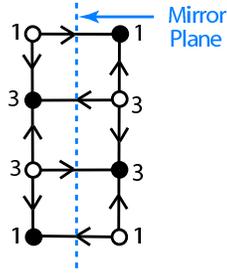
Here we have replaced the labels a, b, c and d with the numerals representing node multiplicity. The top node on the left side of the mirror is identified with the bottom node on the right side, the second node on the left is identified with the third node on the right, and so forth. The mirror is inverting, since the object side is projected upside-down on the image side.

In fact, there is some extra freedom in making these identifications: the image nodes can be identified with the object nodes with a change of sign. There are four consistent ways to arrange this, denoted by including plus signs and minus signs on the mirror plane. Since the right side of the mirror is superfluous, this can be omitted when rendering the adinkra. The four possible multiplets obtained in this way have the following adinkras
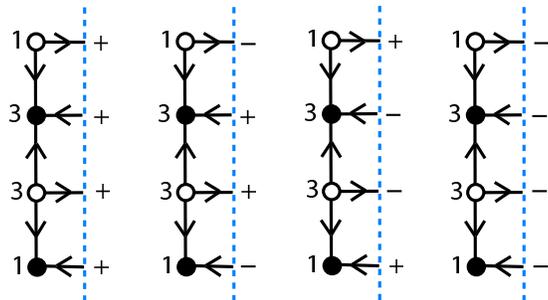
The reason these are the only possibilities is that there is a consistency condition on the placement of the sign flips. Since the image is inverted, it follows that the bottom arrow is a continuation of the top arrow. Similarly the two middle arrows are images of each other. Thus, there are, in essence, only two independent choices of sign flips. The four multiplets obtained in this way describe the four separate scalar multiplets described in [7]. The different sign choices on the mirror plane describe different ways to assign arrow parity to the diagram. These four choices describe different elements of a particular conjugacy class of multiplets, a quaternionic analog of the difference between chiral and antichiral multiplets [9]. The first multiplet shown above is the same

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[9]Different elements of a given conjugacy class are often considered distinct. For instance,

as the chiral multiplet described previously. It is possible to neglect arrow parity and use the undecorated adinkra symbols to describe multiplets as conjugacy classes.

In fact, the manner in which we have organized the discussion of the $N = 4$ scalar multiplets allows for a rigorous proof that the four scalar $N = 4$ scalar multiplets are in fact distinct. The proof relies on the fact that once the $N = 3$ sub-adinkras are given a particular arrow parity then every non-trivial inner automorphism of these representations alters this arrow parity assignment. Inner automorphisms are generated by permutations of nodes and by sign flips. Distinct multiplets are described by adinkras which cannot be mapped into each other by such operations. It is impossible to alter the parity of any arrow which crosses the mirror plane by virtue of node permutations or node sign flips and, at the same time, maintain the intrinsic arrow parity specific to the $N = 3$ cube diagram on either side of the mirror plane. We hope that this discussion may help alleviate skepticism regarding the multiplicity of $N = 4$ scalar multiplets.

There is another interesting way to fold the double-cube diagram shown above. In this maneuver, we pinch the white a node together with the white c node, and at the same time pinch the black a node together with the black c node. The diagram then flattens into the following form
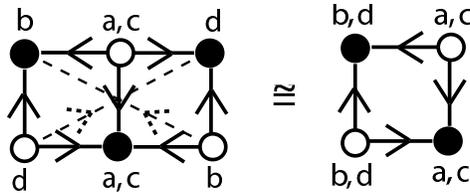

The dotted lines represent the fourth supersymmetry described by the horizontal lines in the double-cube diagram. (Two more dotted lines are coincident with the vertical line connecting the a and c nodes.) We have folded this diagram a final time, by using the middle vertical line as a hinge, lifting the right vertical line out of the page and then placing this on top of the left vertical line. It is clear that the dotted lines land on top of each other with the proper orientation. This is what enables this operation. We are left with a diagram which can be drawn as follows,
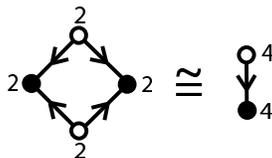
chiral multiplets and antichiral multiplets in supersymmetric field theories describe two distinct elements of a common conjugacy class of representations.

In this form, we see that this adinkra describes a pairwise assembly of the $N = 2$ base adinkras, connected using parallel arrows representing a second pair of supersymmetries. This fully-folded form is more satisfactory for many purposes, as compared to the more complicated forms shown above. But the embedding inside of the root space given above is illuminating and useful in its own right. This multiplet corresponds to the shadow of the $d = 1$ $N = 4$ linear multiplet.

By drawing the analog of the double-cube adinkra for the root multiplet $\Omega_{1+}^{(2)}$ rather than for the base multiplet, and then going through a completely analogous sequence of identifications and folding steps, we arrive at an adinkra symbol which can be drawn as follows,
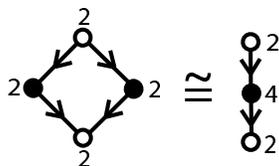
This adinkra is another example of a pairwise assembly of $N = 2$ root multiplets. This time it describes a pair of $\Omega_{1+}^{(2)}$ multiplets rather than a pair of $N = 2$ base multiplets. This adinkra corresponds to the shadow of the $D = 4$ $N = 1$ chiral multiplet. Since we have suppressed arrow parity in this discussion, this more accurately actually describes the conjugacy class corresponding to the chiral multiplet. (Thus, this adinkra also depicts antichiral multiplet, if different choice of arrow parity is selected.)

So far we have explained how the shadows of three of the four irreducible $D = 4$ $N = 1$ multiplets can be described using adinkra symbols. There is one more $D = 4$ $N = 1$ irreducible multiplet, however, the vector multiplet. This is explained in the following subsection.

13.2 Gauge Invariance

Consider the reducible multiplet $\Omega_{6+}^{(4)}$, which is described by the top adinkra in the root tree. This can be drawn in partially-folded and in folded form as follows,
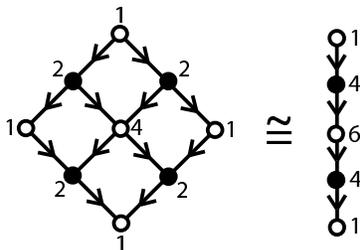
This adinkra exhibits yet another way to reduce degrees of freedom.

A reducible adinkra can be re-defined by adding to it an irreducible adinkra, provided the structure of the smaller adinkra can be layered on top of the larger adinkra such that all arrows line up. In this way, one includes the larger multiplet into a class of multiplets related by a gauge transformation. For example, we notice that chiral adinkra described previously fits onto the topmost diamond inside of the reducible $\Omega_{6+}^{(4)}$ adinkra. We can represent the removal of the associated gauge degree of freedom using the following adinkra calculus,
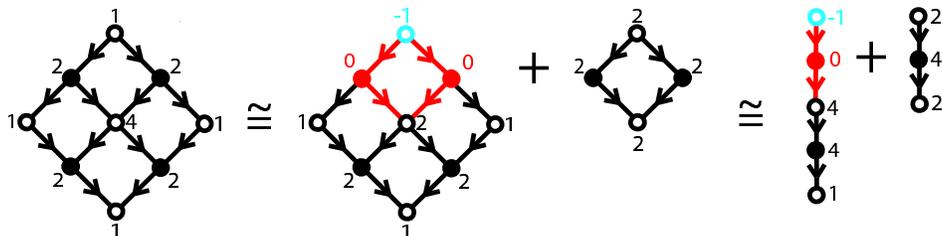

In this diagrammatic equation we see various noteworthy mnemonics at work. First of all, we see how the structure of the chiral multiplet is embedded inside of the reducible vector multiplet. Next, we see how node-by-node the degrees of freedom are subtracted. Most noteworthy of all is the fact that the topmost node has been left with a formally negative field multiplicity. What this means is that one of the two gauge degrees of freedom on the topmost node in the chiral multiplet has been used to remove the single degree of freedom at the topmost node in the vector multiplet. The remaining degree of freedom in the chiral multiplet exists as a residual gauge degree of freedom after all of the possible node subtractions have been performed. The residual gauge degree of freedom then "flows" along the ghost structure as far as possible before finding itself on one of the un-removable nodes. This node then exhibits the gauge degree of freedom in its multiplicity. This demonstrates another rule for locating sub-adinkras which describe embedded gauge structures. Namely, the "flow" of the gauge sub-adinkra must flow "out" of the gauge sub-adinkra onto a non-removable node. This example process is more concisely described in terms of fully-folded adinkras as follows,
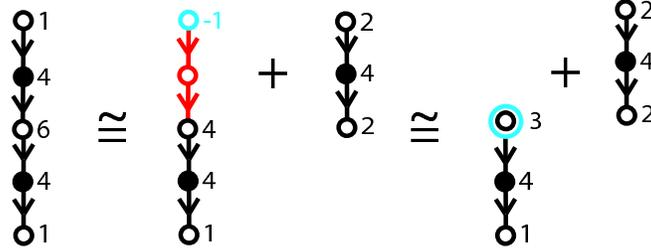
Here the extra circle on the gauge node indicates the ability to perform a gauge shift in the degree of freedom corresponding to this node. This simple example is the diagrammatic representation of the shadow of the well-known Wess-Zumino gauge choice made in the context of $D = 4$ $N = 1$ vector multiplets.

14 Spinning Particles

As a final example, we show how spinning particle multiplets can be described using adinkra symbols. Start with the base adinkra $\Omega_{0+}^{(4)}$, then dualize on the level-two and level-four bosons. This produces the $\Omega_{5+}^{(4)}$ adinkra, given by
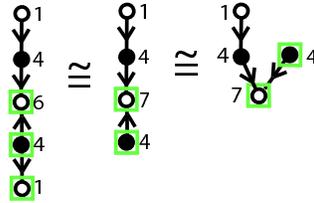
Where in the final step we have oriented the nodes by height, so that all arrows point downward. As a rule, we keep the auxiliary fields separated in adinkra symbols. This adinkra corresponds to the $N = 4$ off-shell spinning particle multiplet first described in [2, 3].

By using a similar process, we can describe the "Universal Spinning Particle Multiplet", also called the USPM, by drawing the base adinkra $\Omega_{0+}^{(N)}$, then dualizing on all bosons except at level-zero. After a sequence of folds, this leads to the USPM adinkra,

,
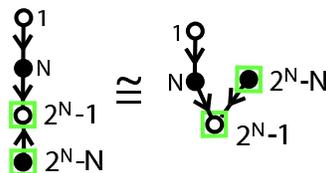
We see that the USPM has $2^N + 2^N$ off-shell degrees of freedom, including $2N - 1$ auxiliary bosons and $2^N - N$ auxiliary fermions.

15   Conclusions

We have described the rudiments of a symbolic method for organizing the representation theory of one-dimensional superalgebras. This relies on special objects, which we have called adinkra symbols, which supply tangible geometric forms to the still-emerging mathematical basis underlying supersymmetry. We are optimistic that these symbols will prove useful in organizing a more rigorous and comprehensive representation theory for off-shell supersymmetry, not just in one-dimension but in higher dimensional field theories as well.

As a demonstration of their power, we have used adinkras to codify, organize and reproduce all known minimal supermultiplets for the cases $N \leq 4$. Building on the concept of root superfields introduced in [1], we have used these symbols to interpret supersymmetry transformations in terms of flows on a corresponding root lattice. At the same time, we have shown how scalar multiplets and reduced chiral multiplets can be explained in terms of embeddings in these lattices, and have given an elegant description of gauge invariance in the case of a reduced Abelian vector multiplets.

We have described a method for altering the appearance of adinkra symbols by folding. This serves several purposes beyond the obvious one, to enable the rendering of multi-dimensional diagrams on a page. Another use for folding adinkras is to allow a topological characterization of supermultiplets. All of the known multiplets fit into the simplest category, described by diagrams which can be folded into a linear chain. The existence of other multiplets, which we have called escheric multiplets, is curious, and we have to wonder what sorts of dynamics might be associated with these.

The adinkra symbol for a given multiplet encodes the corresponding supersymmetry transformation rules. As a consequence, many cumbersome algebraic manipulations characteristic of supersymmetry calculations obtain a fresh look when phrased in terms of these symbols. We wonder if there might be a way to incorporate these adinkras so as to describe superfield dynamics as well. Toward this end, we wonder how our technology should be modified to describe sigma models formulated on curved target spaces.

It has been suggested that *all* supersymmetric theories in all dimensions are connected to each other by different sorts of dualities. The approach to one-dimensional

supersymmetry centered on root superfield technology seem to substantiate this. From this point of view very many multiplets are interconnected by AD maps and by Klein flips. This poses an intriguing dynamical riddle, however. Using a chain of reasoning described in section 3, these duality maps correlate with sigma model target space dualities when an extra central term is switched on in the superalgebra. The riddle is to obtain more comprehensive understanding of the relationships between automorphic dualities and geometric dualities, and to determine the role of supersymmetry central charges in this story. We feel that these observations are hinting at something fundamentally interesting.

Future directions for this investigation include issues pertaining to supersymmetry representation theory and also issues pertaining to dynamics. We intend to generalize the preliminary results described in this paper to include higher values of N, and to establish an understanding for how to "oxidize" one-dimensional multiplets into higher-dimensional multiplets. We would like to use this technology to study supergravity multiplets, and hopefully obtain an aesthetically pleasing alternative way to understand the ad-hoc superfield constraints which plague traditional approaches to this subject. An unapologetic ambition which we have is to use this technology as a step towards finding an off-shell representation of $D = 11$ supergravity.

> *"Mathematics seems to endow one with something like a*
> *new sense."* - Charles Darwin

Appendix: Escheric Multiplets and Central Charges

In this Appendix we provide a few details explaining some of the subtleties associated with the escheric multiplets described in section 10. Although the main presentation

of this paper concerns superalgebras without a central extension, we briefly indicate some connections with such extended superalgebras in this Appendix. Consider the centrally-extended $N = 2$ superalgebra defined by

$$\{\, Q \,,\, Q^\dagger \,\} = H \qquad Q^2 = Z + i\, Y \qquad [\, H \,,\, Z\,] = [\, H \,,\, Y\,] = 0\,, \tag{A.1}$$

where $H = i\, \partial_\tau$ and Z and Y are Hermitian operators which comprise the real and imaginary parts of a supersymmetry central charge. If we define a supersymmetry transformation via $\delta_Q(\epsilon) = \epsilon\, Q + \epsilon^\dagger\, Q^\dagger$, where ϵ is a complex parameter, then (A.1) can be re-written as

$$[\, \delta_Q(\epsilon_1) \,,\, \delta_Q(\epsilon_2)\,] = -2\, i\, \epsilon^\dagger_{[1}\, \epsilon_{2]}\, \partial_t + 2\, (\, \epsilon_1\, \epsilon_2 + \epsilon^\dagger_1\, \epsilon^\dagger_2\,)\, Z - 2\, i\, (\, \epsilon_1\, \epsilon_2 - \epsilon^\dagger_1\, \epsilon^\dagger_2\,)\, Y\,. \tag{A.2}$$

The real part of the central charge Z appears in dimensionally-reduced field theories, where it appears as a shadow of internal momenta modes. The imaginary part of the central charge may have an algebraic connection with conformal supersymmetry. In [4] implications of $Z \neq 0$ were studied, but Y was constrained to vanish.

In section 10 we described two different sorts of duality operations which can be performed on only one of the two level-one nodes in the $\Omega^{(2)}_{2+}$ adinkra, and explained how each of these operations produce new topologically interesting multiplets. In this following two subsections we present the transformation rules for these multiplets, in order to better substantiate the discussion in that section.

A.1 Type I Escherics

If we perform the first sort of duality transformation on the $\Omega^{(2)}_{2+}$ adinkra, by merely reversing the two arrows which connect with one of the two fermionic nodes, we obtain the following dual adinkra,
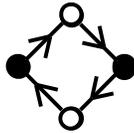
If we write down the transformation rules associated with this adinkra, by following the rules described in section 8.1, we find that the algebra it represents includes a central extension. To see this, first determine the transformation rules from the

diagram using the procedure described above,

$$\delta_Q \phi_1 = i\,\epsilon^1\,\dot{\psi}_1 + i\,\epsilon^2\,\psi_2$$

$$\delta_Q \psi_2 = \epsilon^1\,\phi_2 + \epsilon^2\,\dot{\phi}_1$$

$$\delta_Q \phi_2 = i\,\epsilon^1\,\dot{\psi}_2 - i\,\epsilon^2\,\psi_1$$

$$\delta_Q \psi_1 = \epsilon^1\,\phi_1 - \epsilon^2\,\dot{\phi}_2\,. \tag{A.3}$$

Here we have chosen one of the arrows to have negative parity in order to satisfy the proper sum rule for these parities. If we complexify the supersymmetry parameters by writing $\epsilon = \epsilon^1 + i\,\epsilon^2$ then the algebra satisfied on each of the component fields is the following [10]

$$[\,\delta_Q(\epsilon_1)\,,\,\delta_Q(\epsilon_2)\,] = -2\,i\,\epsilon^\dagger_{[1}\,\epsilon_{2]}\,\partial_t - \tfrac{1}{2}\,i\,(\,\epsilon_1\,\epsilon_2 - \epsilon^\dagger_1\,\epsilon^\dagger_2\,)\,\delta_Y \tag{A.4}$$

where

$$\delta_Y = (\,\partial_t^2 + 1\,)\,. \tag{A.5}$$

Notice that this multiplet includes a purely imaginary central charge, which acts in a non-trivial manner. We refer to escheric multiplets with a non-trivial central charge as "type I" escherics, to distinguish these from the different sorts of multiplets described below.

A.2   Type II Escherics

If we perform the second sort of duality transformation on the $\Omega^{(2)}_{2+}$ adinkra, by writing one of the fermionic nodes as the proper-time derivative of a dual fermion, we obtain transformation rules different than those described in (A.3). In fact, in contrast to that type I escheric multiplet, the transformation rules obtained in this second way do obey the $N = 2$ superalgebra without a central charge. To see this, start with the multiplet $\Omega^{(2)}_{2+}$ and dualize on one of the two fermionic nodes by writing the corresponding fermion field as the proper-time derivative of a dual fermion, which we

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[10]There is a difference between subscripts and superscripts on supersymmetry parameters: subscripts indicate different choices of parameters describing the same supersymmetry whereas superscripts distinguish supersymetries.

will now call ψ_2. This produces the following transformation rules,

$$\delta_Q \psi_1 = -\epsilon^1 \phi_1 + \epsilon^2 \int^t d\tilde{t}\, \phi_2(\tilde{t})$$

$$\delta_Q \phi_1 = -i\,\epsilon^1 \dot{\psi}_1 + i\,\epsilon^2\, \psi_2$$

$$\delta_Q \psi_2 = \epsilon^1 \phi_2 + \epsilon^2 \dot{\phi}_1$$

$$\delta_Q \phi_2 = i\,\epsilon^1 \dot{\psi}_2 + i\,\epsilon^2 \ddot{\psi}_1 . \qquad (A.6)$$

It is easy to check that the algebra (A.1) is satisfied with $Z = Y = 0$ on each of the component fields. These rules can be described by the following adinkra symbol,
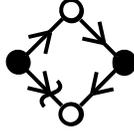
where the newly distinctive type of arrow describes the multiplet $(2,0)_+ \cong (0,2)_+ \cong (0,-1)_- \cong (-1,0)_-$, a sort of $N = 1$ multiplet not described in the main text. The fact that the root labels for this multiplet include integers which are neither 0 nor 1 tell us that this multiplet is not in the root tree. This $N = 1$ multiplet, which has the newly-defined adinkra

has transformation rules

$$\delta_Q \psi = \epsilon \int^t d\tilde{t}\, \phi(\tilde{t})$$

$$\delta_Q \phi = i\,\epsilon \ddot{\psi} . \qquad (A.7)$$

This describes an $N = 1$ supermultiplet which is interestingly distinct from those in the $N = 1$ root tree. The presence of the antiderivatives in (A.6) and (A.7) take on a particular topological significance if the boson which appears under these integrals describes a compact circular dimension. In this case, the antiderivative counts the number of times a particle winds around this circle.

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